PDE for Knockout Barrier option

Math 622

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1 Introduction

In this note to Chapter 7, we consider the risk neutral price for various exotic options:
(i) Knock out Barrier option:

$$V_T = (S_T - K)^+ \mathbf{1}_{\{\max_{[0,T]} S_t \leq b\}}.$$

(ii) Lookback option:

$$V_T = \max_{[0,T]} S_t - S(T).$$

(iii) Asian option:

$$V_T = \Big(\frac{1}{T}\int_0^T S_t dt - K\Big)^+$$

The risk-neutral price $V(t)$ in all of these cases can be expressed as

$$V(t) = \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(T-t)} V_T | \mathcal{F}(t)\Big].$$

It is tempting to write $V(t) = v(t, S(t))$ for some function $v(t,x)$ and start deriving what equation $v(t,x)$ has to satisfy. However, this is incorrect.
Recall that the basis for us to say there exists such a function $v(t,x)$ is because of the Indepndence lemma, which in turns rely on the fact that we can write

$$S_T = S_t \times (\text{ something independent of } \mathcal{F}(t))$$

and we were working with European option, which only depends on S_T.

That is not the case here: all these three exotic options are *path dependent*, i.e. the expression for V_T involves the values of $S_t, 0 \le t \le T$, not just S_T. So apriori, it is not clear that we can find such a $v(t,x)$. Indeed, for the Lookback and Asian option, we will see that the correct function to deal with is $v(t,x,y)$, not $v(t,x)$, where we need to add another component $Y(t)$ to $S(t)$ so that the joint process $S(t), Y(t)$ have the necessary Markov property.
In this section, we will go over how to find the PDE for the Knockout Barrier option.

2 Knock-out Barrier option

Reading material: Ocone's Lecture 4 note part 2, Shreve's Section 7.3.2
Let $S(t)$ satisfies

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Consider the Knock-out Barrier option with barrier b and strike price K:

$$V_T = (S_T - K)^+ \mathbf{1}_{\{\max_{[0,T]} S_t \le b\}}.$$

Note: Necessarily we require $K < b$ and $S(0) < b$ so that $\mathbb{P}(V_T > 0) > 0$.
The risk neutral price $V(t)$ can be written as:

$$V(t) = E\Big[e^{-r(T-t)}(S_T - K)^+ \mathbf{1}_{\{\max_{[0,T]} S_t \le b\}} | \mathcal{F}(t)\Big].$$

We proceed through several steps.
(i) Write $\mathbf{1}_{\{\max_{[0,T]} S_t \le b\}}$ in terms of $S_u, 0 \le u \le t$ and $S_u, t \le u \le T$.
The reason is we want to apply the Independence Lemma (or quote the Markov property of $S(t)$) property of $S(t)$, so heuristically we want to "separate the past and the future". We already know how to do this with S_T. So we apply the same principle to the new term $\mathbf{1}_{\{\max_{[0,T]} S_t \le b\}}$.
This is accomplished as followed:

$$\mathbf{1}_{\{\max_{[0,T]} S_t \le b\}} = \mathbf{1}_{\{\max_{[0,t]} S_u \le b\}} \mathbf{1}_{\{\max_{[t,T]} S_u \le b\}}.$$

It is easy to see why the equality is true: the maximum of the whole path does not exceed b if and only if its maximum on each time interval does not exceed b.
(ii) Recognizing that $\mathbf{1}_{\{\max_{[0,t]} S_u \le b\}} \in \mathcal{F}(t)$, so it can be factored out of $E(.|\mathcal{F}(t))$.

(iii) Define

$$\begin{aligned}\tau_b &:= \inf\{t \geq 0 : S(t) > b\} \wedge T \\ T_b &:= \inf\{t \geq 0 : S(t) = b\} \wedge T\end{aligned}$$

Recall that $P(T_b = \tau_b) = 1$. And so with probability 1:

$$\{\max_{[0,t]} S_u \leq b\} = \{\tau_b \geq t\} = \{T_b \geq t\}.$$

The change from τ_b to T_b might seem unimportant and non-intuitive. But it is to apply the optinal stopping theorem for martingale, see Step (viii).

(iv) Combine (ii) and (iii) we get

$$V(t) = \mathbf{1}_{T_b \geq t} E\Big[e^{-r(T-t)}(S_T - K)^+ \mathbf{1}_{\{\max_{[t,T]} S_u \leq b\}} | \mathcal{F}(t)\Big].$$

(v) Since

$$S(T) = S(t) e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W(T) - W(t))}$$

and

$$\max_{[t,T]} S_u = S_t \max_{[t,T]} e^{(r-\frac{1}{2}\sigma^2)(u-t)+\sigma(W(u)-W(t))},$$

note that $\max_{[t,T]} e^{(r-\frac{1}{2}\sigma^2)(u-t)+\sigma(W(u)-W(t))}$ is independent of $\mathcal{F}(t)$, by the Independence Lemma, we get

$$E\Big[e^{-r(T-t)}(S_T - K)^+ \mathbf{1}_{\{\max_{[t,T]} S_u \leq b\}} | \mathcal{F}(t)\Big] = v(t, S(t)).$$

where

$$v(t,x) := E\Big[e^{-r(T-t)}\big(xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t))} - K\big)^+ \mathbf{1}_{\{x \max_{[t,T]} e^{(r-\frac{1}{2}\sigma^2)(u-t)+\sigma(W(u)-W(t))} \leq b\}}\Big].$$

(vi) (*Crucial point*)

$$V(t) = \mathbf{1}_{T_b \geq t} v(t, S(t)) = v(t, S(t \wedge T_b)).$$

Indeed if $T_b \geq t$ then LHS $= v(t, S(t))$ and $t \wedge T_b = t$ so the RHS $= v(t, S(t))$ and the equality is true.

If $T_b < t$ then LHS $= 0$. $t \wedge T_b = T_b$ so that $S(t \wedge T_b) = b$. Moreover, with probability 1:

$$b \max_{[t,T]} e^{r(u-t)+\sigma(W(u)-W(t))} > b$$

Indeed, if we denote $X(u) := r(u-t) + \sigma(W(u) - W(t)), u \in [t,T]$ then $X(t) = 0$ and by property of Brownian motion,

$$P(X(u) \leq 0, \forall u \in [t,T]) = 0.$$

So there must exist $u \in (t,T]$, $X(u) > 0$ and at that point u, $be^{X(u)} > b$. Thus

$$\begin{aligned} v(t, S(t \wedge T_b)) = v(t,b) &= \Big[e^{-r(T-t)} \big(be^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t))} - K \big)^+ \\ &\quad \mathbf{1}_{\{b \max_{[t,T]} e^{(r-\frac{1}{2}\sigma^2)(u-t)+\sigma(W(u)-W(t))} \leq b\}} \Big] = 0, \end{aligned}$$

and so the RHS $= 0$ as well.

(vii) Another important observation: From the above, we see that the function $v(t,x)$ satisfies $v(t,b) = 0$ for all t. Therefore, it follows that

$$v(\tau, b) = 0,$$

for all stopping time τ taking values in $[0,T]$. From which we derive that

$$v(t \wedge T_b, S_{t \wedge T_b}) = v(t, S_{t \wedge T_b})$$

and

$$e^{-r(t \wedge T_b)} v(t \wedge T_b, S_{t \wedge T_b}) = e^{-rt} v(t, S_{t \wedge T_b}).$$

Indeed, for $t < T_b$ the equalities are clear. For $t > T_b$, then $v(T_b, S_{T_b}) = v(T_b, b) = 0 = v(t,b)$ so the equalities are also true in this case.

This is important because we will apply Ito's formula to $e^{-r(t \wedge T_b)} v(t \wedge T_b, S_{t \wedge T_b})$, not to $e^{-rt} v(t, S_{t \wedge T_b})$. In other words, we want to apply Ito's formula to $e^{-rt} v(t, S_t)$ up to the stopping time T_b only, with the knowledge that $e^{-rt} v(t, S_{t \wedge T_b}) = e^{-rt} V_t$ is a martingale.

Another way to argue is this. We want to apply Ito's formula to $e^{-r(t \wedge T_b)} v(t \wedge T_b, S_{t \wedge T_b})$ so we would want to know that it is a martingale. However, as said above, we only know $e^{-rt} v(t, S_{t \wedge T_b}) = e^{-rt} V_t$ is a martingale. However, applying the fact that a stopped martingale is a martingale, we can also see that indeed $e^{-r(t \wedge T_b)} v(t \wedge T_b, S_{t \wedge T_b})$ is a martingale and our derivation of the PDE below by setting the dt term to 0 is correct.

I believe (constrasting to professor Ocone's note) that we cannot show $e^{-r(t\wedge T_b)}v(t\wedge T_b, S_{t\wedge T_b})$ is a martingale directly by showing say $V_t = v(t\wedge T_b, S_{t\wedge T_b})$. The reason is for $t > T_b$, it is clear that $V_t = 0$. But we do not have a direct interpretation of $v(T_b, B)$ via conditional expectation (see part v for example). So we can only show $V_t = v(t, S_{t\wedge T_b})$ and then stop $e^{-rt}v(t, S_{t\wedge T_b})$ with T_b to achieve a martingale.

(viii) Domain of the PDE:
Since $0 < S(t\wedge T_b) \le b, \forall t \in [0,T]$, we can apply Ito's formula to $e^{-r(t\wedge T_b)}v(t\wedge T_b, S_{t\wedge T_b})$ under the assumption that $v(t,x) \in C^{1,2}$ in the region $[0,T)T_bimes(0,b]$. Thus the domain for our PDE is $[0,T]T_bimes[0,b]$, which is *different* from the domain we used to work on for European call option: $[0,T]\times[0,\infty)$. One of the effect is that we will have *boundary conditions* for our PDE at $x = 0$ and $x = b$.

(ix) Derivation of the PDE:
We have

$$\begin{aligned} S(t\wedge T_b) &= S(0) + \int_0^{t\wedge T_b} rS(u)du + \int_0^{t\wedge T_b} \sigma S(u)dW(u) \\ &= S(0) + \int_0^t \mathbf{1}_{[0,T_b)} rS(u)du + \int_0^t \mathbf{1}_{[0,T_b)}\sigma S(u)dW(u). \end{aligned}$$

Apply Ito's formula to $e^{-r(t\wedge T_b)}v(t\wedge T_b, S_{t\wedge T_b})$ gives

$$\begin{aligned} e^{-rt}v(t, S(t\wedge T_b) &= v(0,S_0) + \int_0^{t\wedge T_b} e^{-ru}\big[-rv + v_t + rS(t)v_x + \frac{1}{2}\sigma^2 S^2(u)v_{xx}\big]du \\ &+ \int_0^{t\wedge T_b} e^{-ru}\sigma S(u)v_x dW_u, \end{aligned}$$

where for all functions v we understood as $v(t,S_t)$, similarly for v_t, v_x, v_{xx}.
Note that this is where the importance of using T_b instead of τ_b is. The stochastic integral

$$\int_0^t e^{-ru}\mathbf{1}_{[0,T_b)}\sigma S(u)v_x dW_u = \int_0^{t\wedge T_b} e^{-ru}\sigma S(u)v_x dW_u$$

is a martingale since T_b is a stopping time. If we use τ_b here we cannot make the same conclusion for technical reason.

Setting the 'dt' term to 0 gives

$$v_t - rv + rxv_x + \frac{1}{2}x^2\sigma^2 v_{xx} = 0, 0 \le t < T, 0 < x < b.$$

Moreover, note that $v(t,0) = 0$ since if $S(t)$ ever hits 0 it will stay there. $v(t,B) = 0$ was explaind in step (vi). These are the boundary conditions for v. We also have the terminal condition $v(T,x) = (x-K)^+$ as usual.
Thus, the PDE that v must satisfy is:

$$\begin{aligned} v_t - rv + rxv_x + \frac{1}{2}x^2\sigma^2 v_{xx} &= 0, 0 \le t < T, 0 < x < b \\ v(t,0) = v(t,b) &= 0 \\ v(T,x) &= (x-K)^+. \end{aligned}$$